PRINCETON UNIVERSITY PHYSICS COMPETITION

GENERAL INFORMATION

Online Part

Guidelines

Student teams will have a total of **one week** to complete the exam from start to finish. We recommend that teams set aside approximately 20+ hours to allow enough time for successful completion. All teams are required to submit their response with a cover page listing the title of their work, the date, the signatures of all contestants on that team, and the team ID number. All other formatting decisions are delegated to the teams themselves, with no one style favored over another. While points will not be deducted for written work, we suggest that teams use a typesetting language (e.g., LaTeX) or a word-processing program (e.g., Microsoft Word/Pages) as unclear answer will not be graded. The packet is divided into smaller sections that will guide participants in understanding the main topic.

Collaboration Policy

Students participating in the competition may only correspond with other members of their team. No other correspondence is allowed, including: mentors, teachers, professors, and other students. While teams are allowed to use a plethora of online resources, participating students are barred from posting content or asking questions related to the exam. As repeated below, teams are also welcome to utilize the Piazza page at `http://piazza.com/princeton_university_physics_competition/fall2015/pupc2015` and ask questions in case something is unclear in the assignment.

Resources

As long as they do not violate the collaboration policy, students have access to the following types of resources:

- **Online**: Teams may use any information they find useful on the Internet. However, under no circumstances may they engage in active interactions such as posting content or asking questions regarding the exam.
- **Piazza page**: Teams are encouraged to create an account in Piazza and register in the class at the following URL:

  `http://piazza.com/princeton_university_physics_competition/fall2015/pupc2015`

  The access code is: **pupc2015p**

  This way, you will be able to ask questions if you'd like to clarify something.
- **Published Materials**: Teams may take advantage of any published material, both printed and/or online.
- **Computational**: Teams may use any computational resources they might find helpful, such as Wolfram Alpha/Mathematica, Matlab, Excel, or lower level programming languages (C++, Java, Python, etc). For some parts of the problem, the use of computational resources is highly advised.

Citations

All student submissions with outside material must include numbered citations. We do not prefer any style of citation in particular. Students may find the following guide useful in learning when to cite sourced material:

`http://www.princeton.edu/pr/pub/integrity/pages/cite/`

Submission

Teams must submit their Online Part solutions by e-mailing **pupc.submit@gmail.com** in accordance with the Test Rules before 23:59 Eastern Time (UTC-5) on Sunday, November 15, 2015. Teams will not be able to submit their solutions to the Online Part at any later time. Regardless of internal formatting, solutions should be submitted as a **single** PDF document with the ".pdf" extension. The e-mail must contain your team ID in the Subject field. Only one person per team, identified as the "team manager" during registration, should send this e-mail. (Team managers will receive their team IDs via e-mail after the Online Part is released, by Sunday, November 8.) Each submitted page should also have on it the team ID number and problem number. Any discrepancies will be dealt with by the current Director of PUPC.

Sponsors

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PUPC 2015: Online Part

Atomic Physics is one of the most studied topics in Physics. Beginning in the late 1800s, it has advanced our understanding of the universe. From the discovery of the atom as the fundamental unit of matter to the nuclear reactor, atomic physics has been growing exponentially for the last century. One of the most important discoveries to date is the use of laser cooling to slow down atoms. The current record of atomic cooling is on the order of pico-Kelvins, or 12 orders of magnitude less than a Kelvin. The method of laser cooling has enabled us to observe many new phenomena. Research in Bose-Einstein condensation, spectroscopy, and even quantum computers requires laser cooling.

Through this competition, students are introduced to various methods in atomic physics centered around the concept of laser cooling. Students will derive concepts from both the semi-classical and quantum mechanical views of atomic theory. Most of the problems require the students to re-derive the given equations, for these problems students are graded from the **work shown in derivation steps**. For some problems that require visualization, students are expected to show clear **scales** on their graph. Problems that require the use of a computational device will always be clearly indicated.

Each section of the background material is followed by several conceptual and/or applied questions that you are expected to answer in any order you choose (while the material is structured in a way that would lead you through the assignment sequence logically). All of the problems here should be able to be solved using only mechanics and electromagnetism, while new concepts will always explained.

These problems cover Nobel prizes beginning with Zeeman in 1902 for the Zeeman Split, continuing with Bohr in 1922 for the Bohr atomic model, Rabi for Rabi oscillations in 1944, Bloch for the optical Bloch equation (and a great last name) in 1952, and Chu, Cohen-Tannoudji, and Phillips in 1997 for laser cooling.

These are very advanced topics, some of which are only taught in advanced undergraduate courses. We expect very few teams to finish the entire packet. We expect that when students find concepts that are unfamiliar, they will be resourceful and research topics using allowed resources that do not violate the collaboration policy. Completing any one of the parts is a major accomplishment for a team, let alone finishing all three parts.

Good luck!

1 Introduction

1.1 Bohr Atom

In 1913 Bohr postulated that the atom can be viewed as a tiny positive charged nucleus, with the electron circulating around it, similar to the earth rotating around the sun. We need to remember that circulating charged material will radiate away its energy; thus, the electron will fall into a lower energy orbit. In his postulate, Bohr suggest that the orbit follows a certain quantum mechanical pattern that we will discuss here. In this section, we will determine what we mean when we say "excited" or "ground" state of an atom.

Consider an atom of hydrogen, consisting of a proton of mass m_p and an electron of mass m_e, in a vacuum and isolated from other atoms. In this problem, we will derive the spectrum of hydrogen in the semi-classical limit.

1. Suppose the electron orbits the proton in a circular orbit of radius r. Write down the forces that the electron experiences.

2. Write down the equations of motion for the system. Do <u>not</u> assume that $m_p \gg m_e$.

3. In this classical derivation we can see that r can assume the value of any real number; thus, the spectrum of the hydrogen atom should be continuous.

   However in 1871 Angström measured 4 discrete spectra of Hydrogen:

Line color	Wavelength (in Ångströms)
red	6562.852 Å
blue-green	4861.33 Å
violet	4340.47 Å
violet	4101.74 Å

Figure 1: Table of hydrogen spectrum observed by Angström

   Bohr postulated in 1913 that the angular momentum of the electron is discrete and given by:

$$L = n\hbar \qquad (n = 1, 2, 3...) \tag{1}$$

   where $\hbar$ is the reduced Planck constant.

   From this assumption, derive the possible energy levels of the electron.

4. From these discrete energy levels, derive the possible radial separations between the proton and the electron r. Show that it will take the form of

$$r = R_E \frac{1}{n^2} \tag{2}$$

   where R_E is a constant. Find R_E in terms of fundamental constants.

5. From this derivation, we can arrive at the formula that was known before Bohr's assumption, known as the Rydberg formula. Remember that the energy of a photon with frequency f is $E_{ph} = hf$. Show that the possible spectra wavelengths λ can be given in the following form:

$$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{3}$$

   for some constant R, with n_f, n_i being positive integers. Find R in terms of fundamental constants.

6. From the Rydberg formula, find the pair n_f, n_i that corresponds to the wavelength λ that Angström observed.

1.2 Zeeman Effect

Before Bohr hypnotized his atomic theory, Dutch physicist Peter Zeeman observed different emission spectra of sodium atoms due to variations in the magnetic field in 1896. For his observation, he was awarded the 1902 Physics Nobel Prize together with Hendrik Lorentz. Initially, it was suspected that the variation is due to the photon interacting with the magnetic field.[1] In this section you will try to derive the weak Zeeman effect for the orbital angular momentum using the Bohr atomic model.

1. Consider a charged ring with linear mass density λ, radius R, and charge Q that is rotating on its normal axis with angular velocity ω. Write down the electromagnetic field at distance r with $r \gg R$ from the center of the ring!

2. Write down the electromagnetic field at a distance r of a magnetic dipole with magnetic moment m located at the origin! We can imagine the magnetic dipole as two oppositely charged q_m separated at distance d, as in Figure 2 with $m = q_m d$, each magnetic charge has field given by

$$B = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}$$

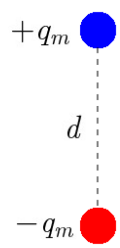

Figure 2: Magnetic charge representation of magnetic dipole

From part 1 and 2 we can see that the electromagnetic field of both systems are identical in a certain limit.

3. Write down this limit!

4. In this limit, we can see that we can replace the magnetic moment m with a form constant multiplied by the angular momentum (L) of the rotating ring, i.e.

$$m = \kappa L \tag{4}$$

   Write down the expression of κ.

5. Do parts 1, 3, and 4 again for a system of a charged sphere with total charge Q, mass density ρ and radius R rotating at angular velocity ω.

6. Comparing the value of κ for both the rotating ring and rotating ball, write down κ in terms of total charge Q and another general variable unrelated to the shape of the object. This constant is known as the **gyromagnetic ratio**.

We know that although magnetic fields cannot produce any work from the Lorentz force, the interaction between two magnetic dipoles can produce some kinetic energy. In order to understand the potential energy

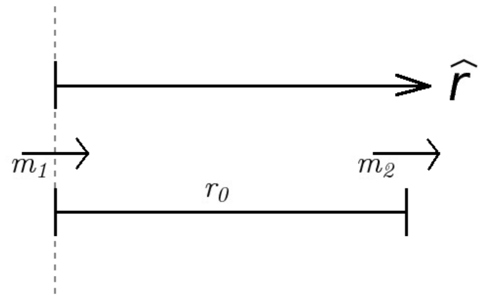

Figure 3: One dimensional system of magnetic dipoles

of a magnetic dipole in a magnetic field, imagine a system drawn in Figure 3.

The system is constrained to move in one dimension while m_1 is held stationary. The system is initially at rest when the distance between dipoles is r_0.

7. Write down the force $\vec{F}$ experienced by m_2 as a function of their separation r.
8. Write down the kinetic energy U of m_2 as a function of their separation r.
9. From the change in kinetic energy, derive the potential energy of the system in this configuration as a function of their separation r.
10. Based on the previous parts, you should be able to write down the general potential energy formulation of a magnetic dipole in a magnetic field. Write down the general expression of U.
11. Using the fact that $\vec{F} = -\vec{\nabla}U$, write down the general expression for the force on a magnetic dipole. Confirm that this agrees with the force that you obtained in the previous part!

1.3 Photon Collisions

We know from physical experiments that we can see light as both an electromagnetic wave and as a particle. In this section, we will investigate this wave-particle duality by treating light as a particle, we can see that it will exert an effective force on atoms due to the absorption and emission. Furthermore, in this section we will also investigate the effect of an atom's velocity on the atoms' perceptions of incoming light, generally known as the relativistic Doppler effect.

The momentum of a photon with wavelength λ is given by the de Broglie relation:

$$p = \frac{h}{\lambda} \tag{5}$$

while at relativistic speed, the momentum of a particle with rest mass m is given by:

$$p_{rel} = \gamma m \beta c \tag{6}$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

[1]"Zeeman effect". Encyclopedia Britannica. Encyclopedia Britannica Online. Encyclopedia Britannica Inc., 2015. Web. 07 Nov. 2015 http://www.britannica.com/science/Zeeman-effect.

1. Prove that the relativistic momentum will reduce to the classical momentum: $p_{rel} \approx p_{cl} = mv$ for $v \ll c$. For what range of v will the classical approximation for momentum differ from the actual, relativistic momentum by 10%?

2. Suppose a photon with wavelength λ collides with an atom with rest mass m that is initially at rest. The atom will absorb the photon and immediately emit it in a random direction. This process can be described by the following collision:

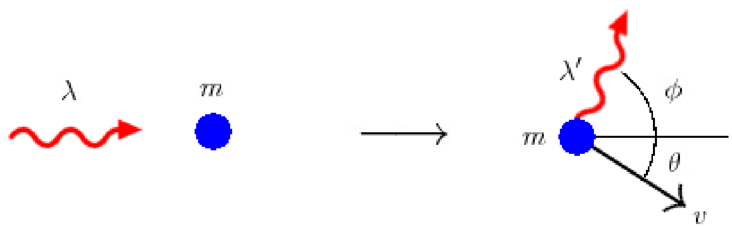

Figure 4: Relativistic collision image for subsection 2

Using conservation of momentum and conservation of energy, derive the velocity v as a function of θ.

3. In this question we will introduce the concept of 4-momentum.

   (a) Prove that, for a relativistic particle of rest mass m:

$$E^2 - (\vec{p}c)^2 = (mc^2)^2 \tag{7}$$

   where $E = \gamma mc^2$

   (b) We can combine the particle energy E and momentum $\vec{p}$ by writing them into a single vector, the 4-momentum:

$$\vec{P} = \left(\frac{E}{c}, \vec{p}\right) = \left(\frac{E}{c}, p_x, p_y, p_z\right) \tag{8}$$

   with the norm of **P** defined as:

$$||\vec{P}||^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2 \tag{9}$$

   Thus, $||\vec{P}||^2 = (mc^2)^2$, which is the same in all reference frames. Therefore $||\vec{P}||^2$ remains invariant under Lorentz transformations.

   Prove that you can recover energy and momentum conservation relations from the conservation of 4-momentum:

$$\sum_{i=1}^{4} \vec{P}_i(t) = \text{Constant} \tag{10}$$

   (c) Now solve part 2 using 4-momentum and verify that you arrive at the same answer as before.

4. What is the minimum/maximum value of v?
5. What is the average value of v (taken over θ)?

We now know what the effect of absorption and spontaneous emission will be on each atom when we treat the photon as a particle. However, we also want to know how the atom views the incoming photon.

6. The atom is moving with the velocity v toward the source of the photon that emits photon with angular frequency ω. In its reference frame, the source is moving with velocity v towards the atom. Using the time dilation relation given by $\Delta t = \gamma \Delta t'$ design a *gedankenexperiment* to measure the relativistic Doppler effect!
7. Show that in limit $\beta \ll 1$ you will find that the answer to the previous part reduces to the the non-relativistic Doppler effect!

2 Atomic States

2.1 Illuminated Atoms

In this section we will consider the statistical aspect of atoms that are illuminated by lasers. We know from the previous section (1.1) that the atom will have at least two possible states, the ground state and the excited state. Throughout the rest of the problem, we will consider atoms with only two possible states, the ground state, which we will denote with sub-index X_g, and the excited state, X_e. In general, as shown in quantum mechanics, it's logical to consider the fraction of atoms in either state, rather than the number of atoms, as we can freely vary the total number of atoms.

In this section we will discuss the use of the semi-classical derivation to explain how the composition of atoms changes due to the incoming photons.

Consider atoms with two energy levels. We denote $|c_e|^2$ to be the fraction of atoms in the excited state and $|c_g|^2$ to be the fraction in the ground state, with

$$|c_e|^2 + |c_g|^2 = 1. \tag{11}$$

For those who are familiar with quantum mechanics, the state of the atoms $|\psi\rangle$ is

$$|\psi\rangle = c_e |e\rangle + c_g |g\rangle . \tag{12}$$

In general, we have that $c_e, c_g \in \mathbb{C}$, and $|c_j|^2 = c_j \overline{c_j}$. Here $\overline{r}$ denotes the complex conjugate where,

$$\begin{aligned} \overline{r} &= \overline{(x+iy)} \\ &= x - iy \ (x, y \in \mathbb{R}) \end{aligned}$$

1. Suppose we shine light on the atoms, such that the number of atoms in the excited state is given by the differential equations

$$i\hbar \frac{dc_g}{dt} = c_e(t)(\hbar\Omega \cos(kz - \omega_l t))e^{-i\omega_a t} \tag{13}$$

$$i\hbar \frac{dc_e}{dt} = c_g(t)(\hbar\Omega \cos(kz - \omega_l t))e^{+i\omega_a t}, \tag{14}$$

where ω_l is the frequency corresponding to the laser frequency and ω_a is the frequency corresponding to the energy levels of the atoms. Ω corresponds to the interaction of the electron from each atom with the electric field amplitude of the photon. Let us define $\delta \equiv \omega_l - \omega_a$. In most experiments, we choose the case where $\delta \ll \omega_e$. Show that in this case the differential equations given above are reduced to

$$\ddot{c_g} - i\delta \dot{c_g} + \frac{\Omega^2}{4} c_g = 0 \tag{15}$$

$$\ddot{c_e} + i\delta \dot{c_e} + \frac{\Omega^2}{4} c_g = 0 \tag{16}$$

2. Solve the separated differential equations in part 1 to get the amplitudes $c_g(t)$ and $c_e(t)$ as functions of time, given the initial conditions:

$$c_g(0) = 1, \quad c_e(0) = 0,$$

   Verify that the amplitudes squared (which gives the probabilities) sum to 1. Furthermore, plot electronically (or sketch by hand) $|c_e|^2$ as a function of time for several values of δ:

   (a) $\delta = 0$
   (b) $\delta = \Omega$
   (c) $\delta = \sqrt{3}\Omega$
   (d) $\delta = 2\sqrt{2}\Omega$

3. It is well known that light is nothing more than electromagnetic radiation. Similar to the Zeeman split that we found in section 1.2, the varying electric field from the photon can introduce a split in the energy level of the atom. This shift is known as the Stark shift and can be calculated by solving the matrix equation:

$$i\hbar \frac{d}{dt}\begin{pmatrix} c'_e \\ c'_g \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 2\delta & \Omega \\ \Omega & 0 \end{pmatrix}\begin{pmatrix} c'_e \\ c'_g \end{pmatrix}$$

   For a certain value of the vector

$$\mathbf{c}_\pm = \begin{pmatrix} c'_{e\pm} \\ c'_{g\pm} \end{pmatrix},$$

   where here $c'_g \equiv c_g$ and $c'_e \equiv c_e e^{-i\delta t}$. It follows that the matrix equation reduces to:

$$i\hbar \frac{d\mathbf{c}_\pm}{dt} = E_\pm \mathbf{c}_\pm.$$

   Find the value of $E_\pm$ and prove that in the limit $\Omega \ll |\delta|$

$$E_+ = \frac{\hbar\Omega^2}{4\delta}, \quad \mathbf{c}_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$E_- = -\hbar\delta - \frac{\hbar\Omega^2}{4\delta}, \quad \mathbf{c}_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

   such that the shift in energy level is

$$\Delta E = \frac{\hbar\Omega^2}{4\delta}$$

4. Show the value of the energy shift in the limit $\Omega \gg |\delta|$!

2.2 Spontaneous Emission

In the previous section we came to understand how the composition of atoms will change due to the incoming photons from a laser. We also need to understand how the atoms will evolve in the absence of photons. We know that an atom can also emit a photon and decay to its ground state. This spontaneous emission case is quite complicated (even Einstein couldn't solve it without simplification), as we have to consider the interactions between the emitted photons and the other atoms. Even worse, the emitted photons can have any polarization and direction. Although the relation is complicated, we can try to solve it by using a simple model that will sum over all possibilities.

From the more detailed quantum mechanical calculation, we will find that the amplitude of the excited state is given by

$$\frac{dc_e}{dt}(t) = -\sum_S |\Omega_S|^2 \int_0^t dt' e^{-i(\omega-\omega_a)(t-t')} c_e(t') \tag{17}$$

Where the subindex S denotes the states of the emitted photon. By using the statistical argument, we can turn the sum into an integral over all possible states. In order to do this we need to go to momentum space!

1. Consider a cube in position space with sides of length L. We want the cube to have fixed boundaries such that the electric field $\vec{E}$ at the boundary is equal to zero. Write down all possible wave vectors $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ that satisfy this boundary condition.

2. We will now move to momentum space! In momentum space, what is the size of the unit volume? Remember that for a polarized photon, we can't differentiate photons on ϕ or $\pi + \phi$ polarization.

3. By using de Broglie relation $k = \frac{\omega}{c}$ and $V = L^3$, write down dn in spherical coordinates. Where n denotes the total number of modes in our momentum space.

4. In order to follow up the calculation, we need to get Ω_S from the energy of interaction between the photon and the electron. This relation is well studied and is given by,

$$\hbar\Omega_S = -\vec{\mu} \cdot \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\hat{\varepsilon} \tag{18}$$

   Where $\hat{\varepsilon}$ is some arbitrary direction of the photon, and $\vec{\mu}$ is related to the atom. We can choose this to point in the $\hat{z}$ direction. We will then be able to replace the summation by integrating over the dn that we found. After integrating over θ and ϕ, show that you will get:

$$\frac{dc_e}{dt}(t) = -\int_0^\infty d\omega\ \omega^3\mu^2 \int_0^t dt' e^{-i(\omega-\omega_a)(t-t')}c_e(t') \tag{19}$$

5. During the derivation of equation 19 we assume that $\vec{\mu}$ varies slowly. In order to solve this integral we need to be able to pull out the coefficient $c_e(t')$ from the time integral. What is the assumption that will make this possible?

6. After we pull out the coefficient $c_e(t)$ we can solve the time integral. Although the time integral is not trivial when we take the limit $t \to \infty$, by using computational software show that we can state the time integral as,

$$\lim_{t\to\infty} \int_0^t dt' e^{-i(\omega-\omega_a)(t-t')} = \pi\delta(\omega - \omega_a) - \mathcal{P}\left(\frac{i}{\omega - \omega_a}\right) \tag{20}$$

   Where δ is the Dirac-delta function and $\mathcal{P}$ is the Cauchy Principal value function.
   This problem is very difficult and will not be worth very many points. You don't need to do this problem to do the rest of the section

7. We can neglect one term of the time integral as it's completely imaginary, and we can integrate over ω. Show that we find

$$\frac{dc_e}{dt} = -\frac{\gamma}{2}c_e \tag{21}$$

   State γ in terms of constants related to the atoms and lasers.

8. The γ that you found here is the true decay rate of the excited state. Why do we say that γ is the true decay rate instead of $\gamma/2$?

2.3 Optical Bloch Calculations

We will now discuss the "drag force" due to photons on an atom. In the actual derivation of "drag force" we can no longer use the semi-classical derivation of the wave function to express how atoms absorb photons. Here we are going to introduce a matrix that will represent the state of the system. Let,

$$\rho = \begin{bmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{bmatrix} \tag{22}$$

When we can clearly differentiate whether the atom is in the excited state or ground state, the density matrix can be related to the coefficients that we found in an earlier part. This is referred to as a "pure state." However, when the atom is in a mix of states, it is referred to as an entangled state, which can not be decomposed. The pure state is given by

$$\rho = \begin{bmatrix} c_e\overline{c_e} & c_e\overline{c_g} \\ c_g\overline{c_e} & c_g\overline{c_g} \end{bmatrix} \tag{23}$$

1. Write down the first order differential equations in terms of Ω and δ that relate each term of the matrices with each other due to the fluctuation in the field from the incoming photons. Use the equations that we have from part 2.1.

The relations between terms in the matrices can be found by observing each term in the pure state representation and the relations that we just found in the previous part. While it's losing it's meaning in an entangled state, the relations are still valid. From part 2.2, we find that due to the spontaneous emission, the coefficient related to the population of the excited state will decay with the rate of $\gamma/2$.

2. Show that due to the spontaneous emission :

$$\left(\frac{d\rho_{ee}}{dt}\right)_{spontaneous} = -\gamma\rho_{ee} \tag{24}$$

3. By combining both effects, show that we will get:

$$\frac{d\rho_{ee}}{dt} = -\gamma\rho_{ee} + \text{Im}\left[\Omega\rho_{ge}e^{-i\delta t}\right] \tag{25}$$

$$\frac{d}{dt}\left(\rho_{ge}e^{-i\delta t}\right) = -\left(\frac{\gamma}{2} + i\delta\right)\rho_{ge}e^{-i\delta t} + \frac{i}{2}\overline{\Omega}\left(\rho_{ee} - \rho_{gg}\right) \tag{26}$$

4. Show the relations between:
   (a) ρ_{ee} and ρ_{gg}
   (b) ρ_{ge} and ρ_{eg}

5. Based on the relation that you found in 4, it is easier to define a new parameter, namely $w \equiv \rho_{gg} - \rho_{ee}$. Using this parameter, derive both $\frac{dw}{dt}$ and $\frac{d\rho_{eg}}{dt}$ in the new variables and other constants!

Based on our initial density matrix of the system, the system will evolve due to the fluctuations in the electromagnetic field from the laser. As in the other physical problem, we are only interested in the steady state solution of the system, as it is hard to reproduce the same system with the exact initial conditions.

6. Write down the equations that represent the steady state of the system!

7. Write down the w and ρ_{eg} in steady state. Express it in the saturation parameter given by

$$s \equiv \frac{|\Omega|^2}{2\left|\frac{\gamma}{2} - i\delta\right|^2} \tag{27}$$

This saturation parameter is related to how well the atoms can absorb the beam. We can also denote s_0 which is the saturation parameter when the laser is at resonance frequency. The resonance saturation parameter is also related to the ratio of the intensity: $s_0 \equiv \frac{I}{I_s}$. Where $I_s \equiv \frac{\pi hc}{3\lambda^3\tau}$. In steady state the absorption and decay rate of atoms are equal and noted by $\gamma_p \equiv \gamma\rho_{ee}$.

8. Write down the expresion of both s_0 and γ_p! Show the limit of γ_p for very high intensities ($I \gg I_s$)!

3 Forces on Atom

3.1 Quantum Mechanic and Classical Physics Relations

From the first section we already know how the photon can transfer its momentum to the atoms from collisions using the semi-classical derivation. While from the second section we know how much the photon will interact with the atoms using Quantum Mechanics. With these tools in our hand we want to derive the effective forces that our atom experiences.

Given that classical phenomena lie in some subset of all natural phenomena, we expect that results from quantum mechanics should reduce to their classical counterparts in some limit. In this subsection, we will explore how expectation values in quantum mechanics obey the classical equations of motion.

1. In quantum mechanics, all physical quantities of interest are represented by linear operators. For example, we have the position operator $\hat{x}$ and the momentum operator $\hat{p}$. Briefly, a linear operator $\hat{\mathcal{O}}$ is a function that takes a vector $|\Psi\rangle$ and returns a new vector $|\Phi\rangle$, which satisfies certain rules of linearity:

$$(\alpha\hat{A} + \beta\hat{B})\,|\Psi\rangle = \alpha(\hat{A}\,|\Psi\rangle) + \beta(\hat{B}\,|\Psi\rangle), \tag{28}$$

where $\alpha, \beta \in \mathbb{C}$ are scalars. To proceed with our derivation, we must define a mathematical operation between linear operators. Given two linear operators $\hat{A}, \hat{B}$, their **commutator bracket** is defined as

$$\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{29}$$

For operators which act as simple multiplication, the commutator bracket is always equal to zero. However, for general linear operators, this is not true. For example, for the operators x^2 (multiply a vector/function by x^2) and $\partial/\partial x$ (take the derivative of the vector/function with respect to x).

$$\begin{aligned}\left[x^2, \frac{\partial}{\partial x}\right] f &= x^2\frac{\partial}{\partial x}f - \frac{\partial}{\partial x}(x^2 f)\\ &= x^2\frac{\partial f}{\partial x} - 2xf - x^2\frac{\partial f}{\partial x}\\ &= -2xf\end{aligned}$$

Thus we find,

$$\left[x, \frac{\partial}{\partial x}\right] = 2x \tag{30}$$

For the momentum operator, defined as $\hat{p} \equiv -i\hbar\frac{\partial}{\partial x}$, and Hamiltonian (Energy Operator) $\hat{H}$, prove that

$$\left[\hat{H}, \hat{p}\right] = i\hbar\frac{\partial \hat{H}}{\partial x} \tag{31}$$

2. The expectation value in quantum mechanics is defined as the expected outcome of measurements on a system. It is mathematically defined as:

$$\left\langle \hat{Q} \right\rangle = \langle \Psi | \, \hat{Q} \, | \Psi \rangle \tag{32}$$

Where here $|\Psi\rangle$ is the wave function of our system, and $\langle \Psi|$ is defined as the conjugate transpose of the wave function. In basis vector notation this is noted as,

$$\langle \Psi | = | \Psi \rangle^{\dagger}$$
$$\begin{pmatrix} \overline{a} & \overline{b} & \overline{c} \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^{\dagger}$$

Given that the Hamiltonian operator is related to the time evolution of the Schrödinger equation,

$$i\hbar \frac{d}{dt} | \Psi \rangle = \hat{H} \, | \Psi \rangle \tag{33}$$

Show that,

$$i\hbar \frac{d}{dt} \langle \Psi | = -\hat{H} \, \langle \Psi | \tag{34}$$

3. Prove that for any operator $\hat{Q}$, the time evolution of it's expectation value is given by,

$$\frac{d}{dt} \left\langle \hat{Q} \right\rangle = \frac{i}{\hbar} \left\langle \left[\hat{H}, \hat{Q} \right] \right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle \tag{35}$$

4. Using equation 35 prove the Ehrenfest Theorem that states,

$$F = -\left\langle \frac{\partial \hat{V}}{\partial t} \right\rangle, \tag{36}$$

where the expectated force is defined as

$$F = \frac{d \langle \hat{p} \rangle}{dt} \tag{37}$$

and the Hamiltonian operator is:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V} \tag{38}$$

where $\hat{V}$ is the potential energy operator of the system. From this theorem we can see that the expectation value of the force obey the classical law $F = -\frac{\partial V}{\partial x}$.

3.2 Cooling

We are now ready to derive the force on an atom due to interactions with photons from the laser as we have already derived both statistical and quantum mechanical aspects of the interaction between photons and atoms.

1. We will first consider the effect of absorption followed by spontaneous emission by the atoms. Consider a beam of photons with incoming wave vector $\vec{k}$. In this first estimation assume that the atoms are not interacting with each other. Calculate the force experienced by the atoms in steady state!
Hint: What is the average of momentum change for each spontaneous emission? Remember that the direction is random.

2. The first estimation that we do in 1 is not sufficient for the system where the atoms are interacting with each other. From quantum mechanics we can get that the actual force experienced by the group of atoms is given by,

$$F = \left(\frac{\partial \Omega}{\partial x} \overline{\rho_{eg}} - \frac{\partial \overline{\Omega}}{\partial x} \rho_{eg} \right) \tag{39}$$

Using the fact that we can state the position derivative of Ω as

$$\frac{\partial \Omega}{\partial x} = (q_r + i q_i)\Omega, \tag{40}$$

derive the equations for $\frac{d\Omega}{dt}$, $\frac{dw}{dt}$, and $\frac{d\rho_{eg}}{dt}$ for the first order correction of the steady state condition that we initially found in 6!
Hint: Use the steady state value of w and ρ_{eg} for deriving all of these quantities as we only consider the first order corection.

We will now consider atoms trapped in a region with standing waves of photons.

3. Prove that in standing waves $q_i = 0$ and $q_r = -k \tan(kx)$!

4. Derive the expression of F in this case, using equation 39 and other quantities that we found earlier in part 2!

5. Show that, in the limit $s \ll 1$, we can represent the force as

$$F(x) = F_0(x) - \beta(x) v \tag{41}$$

6. By averaging the force over x, show that $\langle F_0(x) \rangle = 0$, thus the effective force is equal to $\langle F \rangle = -\beta x$ where $\beta \equiv \langle \beta(x) \rangle$.

7. Write down the expression for β.

8. In the actual experiment, we can determine the value of β as a function of detuning to determine the saturation parameter on the resonance frequency s_0 by changing the detuning of our laser. Using the data given below in Figure 5, determine s_0 of our atoms if we are using NdCR:YAG laser with wavelength of 1.064 μm.

   Plot the data, and the linearized data! From the regression, estimate the value of s_0 and give error estimates as well!
   Hint: You should be able to do simple linear regression on the equation of β

In the previous part, we were calculating the drag force on atoms suspended in standing waves. We see that the atoms will experience a drag force proportional to the velocity. Experimentally, it is harder to create a cavity with standing waves of photon. The usual procedure of laser cooling involves two lasers opposite to each other with the same frequency ω_l. Due to the Doppler shift, the atoms see the two lasers as having different wavelengths. When the intensity of the laser is low enough, we can assume that the atoms will not interact with each other, thus we can use the steady state force we found in 1.

9. Determine both F_+ and F_- experienced by the atoms! Where F_+ is the force on atoms due to the red-shifted laser.

10. Write down the total force $F_{OM} = F_+ + F_-$ experienced by the atoms. In limit of small Doppler shift, show that we can write this force as

$$F_{OM} = -\eta v. \tag{42}$$

11. What is the requirement of ω_l such that the atoms are decelerating?

$\frac{\delta}{\gamma}$	$-\beta$ (10^{-81} kg/s)
0.	0.955427
0.1	33.3342
0.2	48.1505
0.3	54.555
0.4	45.583
0.5	38.2596
0.6	33.1837
0.7	26.5604
0.8	21.1309
0.9	16.4718
1.	15.8253
1.1	9.23675
1.2	9.76491
1.3	5.82267
1.4	8.03579
1.5	8.16083
1.6	0.639299
1.7	3.21542
1.8	5.3675
1.9	2.41691
2.	1.17766
2.1	5.42204
2.2	3.85394
2.3	1.34887
2.4	2.08414
2.5	1.75254
2.6	1.02977
2.7	1.48006
2.8	-0.187162
2.9	-2.51977
3.	0.285455

Figure 5: Table for subsection 8

12. Show that in the limit $s_0 \ll 1$ we will find that $\eta = \beta$!
13. Now we no longer assume the small Dopper shift. Determine the maximum of F_{OM} as a function of v!

By knowing the maximum F_{OM} we will know how to set up our lasers to achieve the maximum deceleration.

3.3 Laser Cooling Limit

While laser cooling can slow down the particle, we know that there exists a limit to this method. In this subsection, you are asked to discuss this limitation and how we should approach it.

1. Discuss the source of this limit!
   Hint: Consider an atom initially at rest and experiencing the absorption - emission process
2. Show that the heating power due to the phenomena in 1 is given by,

$$\Phi_{heat} = 2\frac{\hbar k^2}{M}\gamma_p \tag{43}$$

3. Derive the minimum temperature of the system as a function of ω_l!
   Hint: Use the F_{OM}
4. Show that there is a universal limit (even when we already minimize the ω_l) that only depends on the decay rate. This is known as $T_{Doppler}$.

End of the Paper