Relativistic Electrodynamics Section

The subject matter of this section is relativistic electrodynamics, that is, the study of electricity and magnetism incorporating Einstein’s Special Theory of Relativity. The goal of this topic is to give students a feel for the beauty of Maxwell’s equations (the core equations of electricity and magnetism) when formulated in relativistic language and to give them an understanding of unification of these two seemingly different subjects into a grander electromagnetic field.

In introductory electricity and magnetism classes, students typically learn about electric fields and magnetic fields and perhaps learn about the basics of how the two are connected, but such courses tend not to give the student an understanding of the deep connection between these two fundamental subjects. In particular, students miss out on the fact that electricity and magnetism are not two different phenomena, but rather part of the same overarching mathematical structure (the electromagnetic field tensor, which will be introduced in this section). In fact, what one calls magnetism can be viewed as manifestation of a moving electric field and vice versa. Furthermore, while students are introduced to Maxwell’s equations, they are not shown how they can be condensed into a single fundamental equation with the language of four vectors. This section teach students how to do this, offering derivations and problems along the way. Our medium will be relativity, so that will be introduced first. Then we will get into reformulating electrodynamics.

Contents

1 Basics of Special Relativity
   1.1 Conceptual Basics of Special Relativity ................................................. 2
   1.2 Introduction to Four Vectors ................................................................. 2
   1.3 Einstein Summation ............................................................................... 3
   1.4 The Spacetime Interval .......................................................................... 4
   1.5 Mechanics in the language of four-vectors .............................................. 4

2 Relativistic Electrodynamics and Tensors ................................................... 6
   2.1 Four-dimensional Calculus ................................................................. 6
   2.2 Four-current ......................................................................................... 6
   2.3 Four-Potential ....................................................................................... 7
   2.4 The Electromagnetic Field Tensor ....................................................... 8
   2.5 The Transformations of the Fields ....................................................... 8
   2.6 Field Transformation Problems ......................................................... 9

Learning goals of this topic: As stated above, the goal of this topic is to give students an appreciation for the beauty of the relativistic formulation of electricity and magnetism and an understanding of the deep relationship between the subjects.

Topic format: This document consists of explanatory sections with helpful exercises and questions interspersed. Much of the grading will be based on sections that ask you to explain or interpret results in your own words. We are looking to see how well you understand the subject and are less concerned with minor errors in completing exercises.

Some sections contain no questions. We would encourage you not to skip reading these parts, as all sections will be beneficial for your understanding. If your first read-through leaves you perplexed, baffled, or utterly nonplussed, don’t be discouraged: these topics take time to absorb!
1 Basics of Special Relativity

1.1 Conceptual Basics of Special Relativity

Perhaps the most exciting aspect of learning special relativity is having your physical intuition turned on its head and sent through a meat grinder. Please refer to this webpage\textsuperscript{1} to get a conceptual background for special relativity, then answer the questions below. You may skip the sections on World Lines and Light Cones.

1.1.1 Problem: The Barn Paradox

This is a classic problem in special relativity; it will be up to you, however, to determine whether or not it is truly deserving of the label “paradox”.

In Fig. 1, a runner is carrying a pole that is slightly longer than the length of a barn. The barn in question has two doors; one in the front and one in the back. The runner is capable of running at speeds comparable to the speed of light, and wants to know if she can fit the length of the pole inside the barn in a single instant; to test this, a farmer standing outside the barn will close both of the doors at the same time, and then open them again immediately after (you may assume the doors take only an infinitesimal length of time to open and close, and remain closed for a negligible amount of time). Is there any way the pole will fit inside the barn? Describe the events of the runner entering the barn, the doors closing, and the runner leaving the barn from the point of view of the runner and of the farmer. Is this really a “paradox”? Why or why not?

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Figure1.png}
\caption{The Pole in the Barn Paradox}
\end{figure}

\textsuperscript{1}Copy and paste \url{http://abyss.uoregon.edu/21st_century_science/lectures/lec06.html} into your browser if the link doesn’t work.
1.2 Introduction to Four Vectors

Up until now, you have likely done most or all of your physics with three-vectors, collections of three numbers which represent some physical quantity, such as position, momentum, force, some field, etc. In special relativity, it will be more useful to extend this concept to that of a four-vector.

To introduce four-vectors, consider the collection of coordinates $dx^\mu = (dx^0, dx^1, dx^2, dx^3) = (ct, dx, dy, dz)^2$

Consider two frames--S and S’ (see Fig.2). S is stationary and S’ moves in the positive x-direction with velocity v relative to S. The spacetime coordinates in S are given by $(ct, dx, dy, dz)$ and those in S’ by $(ct', dx', dy', dz')$.

According to special relativity, the coordinates are related by the Lorentz transformation

\[
\begin{align*}
ct' &= \gamma (ct - \beta dx) \\
 dx' &= \gamma (dx - \beta cd) \\
 dy' &= dy \\
dz' &= dz,
\end{align*}
\]

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$.

Or, in matrix form,

\[
\begin{pmatrix}
ct' \\
dx' \\
dy' \\
dz'
\end{pmatrix} = \begin{pmatrix}
\gamma & -\gamma \beta & 0 & 0 \\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
ct \\
dx \\
dy \\
dz
\end{pmatrix}.
\]

A transformation between inertial frames such as this is known as a boost. Unless otherwise noted, the remainder of the problems in this portion of the exam will refer to boosts.\(^4\)

$dx^\mu = (ct, dx, dy, dz)$, and any other collection of four-coordinates which obey the Lorentz transformation under a boost, are known as four-vectors.

\(^2\)This superscript notation is new. Essentially, we are saying that $dx^\mu$ represents the whole vector, since you can reproduce the entire vector by plugging in each of the different values of $\mu$ (from 0-3).

\(^3\)These can be derived directly from Einstein’s postulates and an assumption of linearity; however it will be more convenient to simply start with them for our purposes. Furthermore, the velocity can be in any direction. We have used the x-direction for simplicity.

\(^4\)Lorentz transformations also include regular spatial rotations. And a four vector can also be subject to translation. However, these will not be explored during this section of the exam.
1.3 Einstein Summation

Einstein summation is typically used for notational brevity; instead of writing out a sum each time we want to refer to a quantity, we can simply write out two terms with indices. When you see two quantities multiplied together, multiply the $i$th term from each quantity and sum over all indices.

In other words,

$$a_\mu b^\mu = \sum_{i=\mu}^n a_\mu b^\mu;$$

here, $\mu$ represents the index and both quantities are four-vectors. Notice one index is a superscript and the other is a subscript; the quantity with a superscript has vertical vector components, and the quantity with the subscript has horizontal vector components.\(^5\) For the purposes of this section, $n$ is equal to 4; this term represents the dimension of the space.

1.4 The Spacetime Interval

Consider the collection of coordinates $dx^\mu$ in the frame $S$ from section 1.1. We have coordinates $(cdt, dx, dy, dz)$, and since the index is noted as a superscript, we know that this is a column vector (called a “contravariant vector”). If we wanted to change our contravariant vector to a row vector (covariant vector), we simply make the first component negative, and the vector becomes $dx_\mu$ with coordinates $(−cdt, dx, dy, dz)$.\(^6\)

We can compute the spacetime interval (the squared length of a differential amount of spacetime) by

$$ds^2 = dx_\mu dx^\mu.$$  

This is a four-dimensional dot product, and $ds$ is the norm of the displacement four-vector.

1.4.1 Problem: Invariance of the Spacetime Interval

Expand the above product and show that it is invariant under Lorentz transformations.

1.4.2 Problem: Time Dilation

Using the spacetime interval, compute the relation between (differential) time in the unprimed frame and the primed frame.\(^7\) This is known as time dilation. Specifically, what does this imply in terms of how quickly time passes between two frames? Give a real-world example of time dilation in action (you may greatly exaggerate the effects).

1.4.3 Problem: Length Contraction

Compute the relation between length in the unprimed and primed frames. This is the other half of the

\(^5\)This may look a little familiar if you’ve had a lot of practice with the dot product in more mathematical applications; if not, you needn’t worry too much about this detail.

\(^6\)In fact, we are changing from a contravariant to a covariant vector via the metric of the space (the Minkowski metric in special relativity). However, we will not go into those details in this portion of the exam.

\(^7\)The primed frame is known as the proper frame. A particle which moves with a certain velocity in the unprimed frame is stationary in this frame.
special relativity puzzle, known as length contraction. Give an example of the relationship between time
dilation and length contraction; in other words, give an example of a scenario in which different observers
will experience one or the other of these effects. Be sure to explain your reasoning. *(Hint: you may not use
the barn problem from above, but it can serve as a good starting point to come up with your own scenario.)*

1.4.4 Problem: Relativity and Rotations

The boost transformation law is typically derived by shifting between inertial frames and using the postulates
of relativity, as well as an assumption of linearity. However, one can also derive them by a rotation. In 3-
space, a rotation by angle $\theta$ about the z-axis transforms the coordinates as

$$
x' = x \cos \theta + y \sin \theta
$$

$$
y' = -x \sin \theta + y \cos \theta
$$

$$
z' = z
$$

Note that this rotation preserves length $ds = \sqrt{x'^2 + y'^2 + z'^2}$.

Using the idea that the Lorentz transformation must preserve the spacetime interval, choose a system of
four coordinates and use the rotation law above (rotation of two axes about a third) to derive the Lorentz
transformation. *(Hint: use the relation between $\gamma$ and $\beta$ to define the angle of rotation.)*

1.5 Mechanics in the language of four-vectors

Now that you have been introduced to four-vectors, let’s formulate all of mechanics in terms of them.

The four-velocity is given by: $u^\mu = \frac{dx^\mu}{d\tau}$.

Here $d\tau$ denotes proper time, which is the time as measured by an observer moving in S’ (the moving
frame). In short, $d\tau = dt'$.

The four-momentum by: $p^\mu = m_0 u^\mu$, where $m_0$ is the rest mass (mass in a stationery frame).

And the four-force by: $F^\mu = \frac{dp^\mu}{d\tau}$.

1.5.1 Problem: Four-Velocity

a) Why does one have to differentiate with respect to proper time? What happens if you don’t?

b) Derive the Einstein velocity addition laws from the Lorentz transformation for the four-velocity.

These laws are:

$$
u = \frac{v + u'}{1 + \frac{vu'}{c^2}}
$$

$$
u' = \frac{u - v}{1 - \frac{vu}{c^2}}.
$$
1.5.2 Problem: Invariance of Energy and Momentum

The momentum four-vector is given by \( p = (E, p_x, p_y, p_z) \). Use this information to derive the relationship between energy and momentum.

1.5.3 Problem: Four-Acceleration

Prove that the dot product of four-acceleration and four-velocity is zero for any arbitrary mass \( m \).

2 Relativistic Electrodynamics and Tensors

2.1 Four-dimensional Calculus

One can extend the gradient of three dimensions to four as

\[ \partial_\mu = (\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}) \]

The divergence can then be defined as the (four-dimensional) dot product of this gradient with an arbitrary four-vector.

A further operator, the d’Alembertian, may be defined as follows:

\[ \Box^2 \equiv \frac{1}{c^2} \partial^2 - \nabla^2 \], where \( \nabla^2 \) is the ordinary Laplacian.

2.2 Four-Current

Using the four-velocity, we can define a current density\(^8\) four-vector as

\[ j^\mu \equiv \rho_0 u^\mu \].

2.2.1 Problem: The Continuity Equation

a) Prove, that in 3 dimensions,

\[ \nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t} \]

This is known as the Continuity Equation. \( \text{Hint: consider a small sphere with charge flowing outward and use the fact that charge is conserved.} \)

b) Now show that, with four-vectors, you can express this neatly as

\[ \partial_\mu j^\mu = 0 \].

What does this mean in terms of the four-dimensional divergence?

\(^8\)In 3-dimensions, current density is defined as \( \vec{j} = \rho \vec{u} \).
2.3 Four-Potential

In terms of 3-vectors, Maxwell’s equations (the core equations of electricity and magnetism) read

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]
\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
\[ e^2 \nabla \times \vec{B} = \frac{j}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t} \]

2.3.1 Problem: Maxwell’s Equations in Terms of the Potentials

a) From these equations, show that \( \vec{B} \) and \( \vec{E} \) may be written as

\[ \vec{B} = \nabla \times \vec{A} \]
\[ \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}, \]

where \( \vec{A} \) is known as the vector potential and \( \phi \) is known as the scalar potential. Specifically, which of Maxwell’s equations did you use to attain these equations and which properties of vector calculus were used?

b) Using the conditions from a), show that one may write Maxwell’s equations as

\[ \Box^2 \left( \frac{\phi}{c} \right) = \frac{c \rho}{\mu_0} \]
\[ \Box^2 \vec{A} = \frac{j}{\mu_0} \]

You may find the relation \( \nabla \times (\nabla \times \vec{C}) = \nabla (\nabla \cdot \vec{C}) - \nabla^2 \vec{C} \), where \( \vec{C} \) is some vector, useful.

Note: In this derivation, it will be necessary to exercise something known as gauge freedom, which means that you will have to choose a value for \( \nabla \cdot \vec{A} \). This choice can be anything, so long as it helps you to attain the form of the equations given above. Why did you make this choice? How did you know it was the proper choice to simplify the equations?

c) Show that \( A^\mu = (\frac{\phi}{c}, \vec{A}) \) is a four-vector by

i. showing that \( \Box^2 \) is Lorentz invariant. Explain why is this condition sufficient show that \( A^\mu \) is a four-vector.

ii. by showing \( dV/r \) is Lorentz invariant. Why is this condition sufficient? It may help to recall that

\[ \phi = \int \frac{d\phi}{\gamma} \quad \text{and} \quad \vec{A} = \int \frac{\vec{E}}{\gamma} dV. \]

d) Using the fact that \( A^\mu \) is a four-vector, show that all of Maxwell’s equations may be written as

\[ \Box^2 A^\mu = \frac{j^\mu}{\mu_0}. \]

Be sure to carefully explain your logic in getting from the equations of part b) to here.
2.3.2 Problem: Forces in Different Frames

An electric force $F_e$ acts on a particle of mass $m$ and charge $q^+$ that is moving at a velocity $v$ in the unprimed frame. Find $F'$, the same force in the particle’s rest frame.

2.3.3 Problem: Particles in a Wire

Refer to the Fig. 3 above. In the lab frame, negative particles are moving in a wire at a speed $u$. The positive particles are stationary. The positive particles have a charge density $\lambda^+$, and the negative particles have a charge density $\lambda^-$. There is another particle of charge $q^+$ a distance $r$ away from the wire, moving in the same direction as the negative particles in the wire at a speed $v$.

a) Find the total force on the particle of charge $q^+$ in the lab frame.

b) Find the total force on the same particle in its own rest frame. What does this say about the nature of electric and magnetic forces? Use your answers from the previous parts of this question in your explanation.

2.4 The Electromagnetic Field Tensor

a) Using the equations from 3.3.1a, write the components of $\vec{E}$ and $\vec{B}$ in terms of the components of $A_\mu$ and $\partial_\mu$, using $A_\mu = (A_0, A_1, A_2, A_3) = (\phi, A_x, A_y, A_z)$ and $\partial_\mu = (\partial_0, \partial_1, \partial_2, \partial_3) = (1/c \partial/t, \partial/x, \partial/y, \partial/z)$.

b) Using these examples, write a general expression for a field component in terms of $A_\mu$ and $\partial_\mu$, which will be a function of two indices, $\mu$ and $\nu$. Call this $T_{\mu\nu}$. This is known as the electromagnetic field tensor.

c) What happens when $\mu = \nu$ and when the two indices are flipped? How many entries does this account for? How many entries does $T_{\mu\nu}$ have in total?

d) Now write down $T_{\mu\nu}$ as a matrix.

2.5 The Transformations of the Fields

The reason the matrix above is so crucial is because of how it transforms. It is a rank 2 tensor and so transforms in an understandable way under a boost. Applying this transformation gives us how the electric and magnetic fields change under a boost.
a) The Lorentz transformation for the electric and magnetic fields is given in tensor language as

\[ T'_{\mu\nu} = \Lambda T_{\mu\nu} \Lambda^t, \]

where \( T_{\mu\nu} \) is the field tensor, \( \Lambda \) denotes the Lorentz transformation matrix from earlier and \( t \) denotes transpose. From this relation, show how the electric and magnetic fields transform under a boost.

More generally, the fields transform as

\[ \vec{E}'_\parallel = \vec{E}_\parallel, \]
\[ \vec{B}'_\parallel = \vec{B}_\parallel, \]
\[ \vec{E}'_\perp = \gamma (\vec{E}_\perp + \vec{v} \times \vec{B}), \]
\[ \vec{B}'_\perp = \gamma (\vec{B}_\perp - \frac{1}{c^2} \vec{v} \times \vec{E}). \]

b) Using Coulomb’s law for the electrostatic field of a point charge, \( \vec{E} = \frac{q}{4\pi\varepsilon_0 r^3} \vec{r} \), and the transformation laws above, derive the law for the magnetic field produced by a moving charge. What does this tell you about the relation between electric and magnetic fields? This is known as the Biot-Savart law for point charges. (It may easily be extended to compute the magnetic field of currents.)

c) Now, using the Biot-Savart law for point charges, and the field transformations above, derive Coulomb’s law (take the limit of small velocities). What does this tell you about the relation between electric and magnetic fields?

### 2.6 Field Transformation Problems

#### 2.6.1 Problem: Moving Solenoid

A long solenoid with \( N \) turns is positioned in frame \( F \), as shown in Fig. 4 above. If the solenoid moves in the positive \( x \) direction at a velocity \( v \), find the electric and magnetic fields inside the solenoid in frame \( F \) and in the solenoid’s rest frame.

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9 The parallel and perpendicular signs are in reference to direction of motion; i.e., if a particle is moving parallel to electric field lines, use the equations for \( \vec{E}_\parallel \).
2.6.2 Correction for Maxwell’s Equations

We have shown that one of Maxwells equations is \( c^2 \nabla \times \vec{B} = \frac{\vec{J}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t} \). You may have also seen it written (incorrectly) as \( \nabla \times \vec{B} = uJ \). This is a simplified version of the equation that is sufficient for a lot of uses of the equation. Provide an example in which the \( \frac{\partial \vec{E}}{\partial t} \) component is necessary in order to complete the problem.